Signals and Systems
E-623

Lecture 10
Differential Equation Modeling
(Electrical – Mechanical - Fluid)
& Analogous Systems

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Differential Equation Models

Continuous-time systems are often specified by an input/output differential equation that can be generated by application of the laws of physics.

What is meant by analogous systems?

An analogous electrical and mechanical system will have differential equations of the same form.

Differential Equation Models

2.2 MECHANICAL SYSTEMS

Mechanical systems are of two kinds:

1. Translational system which consists of mass, spring, dash-pot and lever arrangement.
2. Rotational system which consists of inertia, torsional spring, dash-pot and gear arrangement.

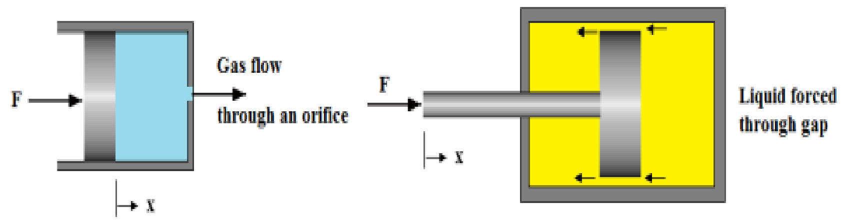
2.2.1 Mechanical Translational Systems

1. $f(t)$ = Applied force (input), (N).
2. M = Mass, (Kg).
3. B = Viscous friction coefficient of dash-pot, (N/m/sec)
4. K = Spring stiffness constant, (N/m).
5. x = Displacement, (m).
6. $v = \frac{dx}{dt}$ = Linear velocity, (m/sec).
7. $a = \frac{d^2x}{dt^2}$ = Linear acceleration, (m/sec^2).

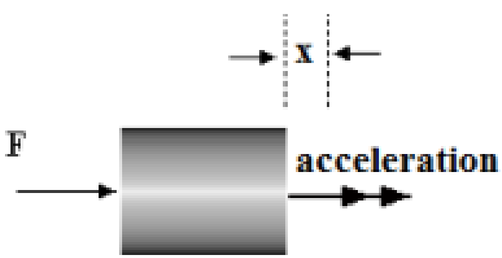
Mechanical Transitional Systems

In mechanical translational system, mass is the element which stores kinetic energy, linear spring stores potential energy and dash-pot dissipates energy and provides damping to the system.

Damper = Dashpot



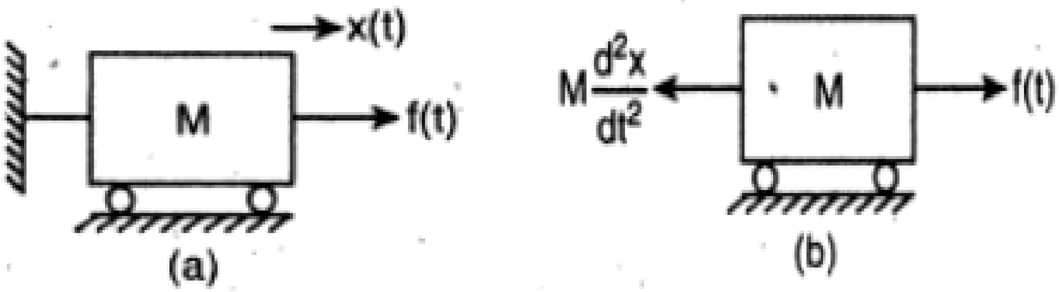
Spring



Mass

1. Mass

When a force $f(t)$ is applied to a mass M, it stores kinetic energy and develops an opposing force $= M\frac{d^2x}{dt^2}$ which acts in the direction opposite to the applied force as shown



The free body diagram for mass M

According to Newton's law of motion, the sum of the applied forces is equal to the sum of the reaction forces which act opposite to that of the applied forces.

$$f(t) = M\frac{d^2x}{dt^2} \qquad \text{acceleration} = \frac{d^2x}{dt^2}$$

Changed into Laplace form: $F = Ms^2 x$

Mechanical Transitional Systems

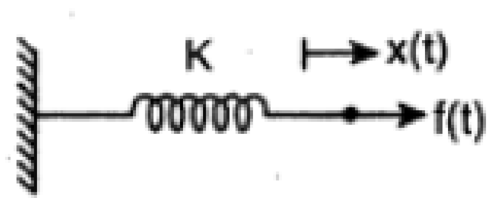
2. Spring

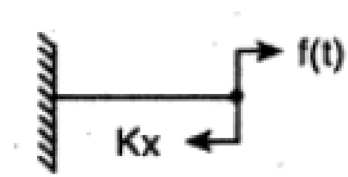
- A spring element is one that stores energy due to the elastic deformation that results from the application of a force.
- Over its linear region, the spring satisfies Hook's law that relates the force to the displacement by the expression:

$$f(t) = Kx(t) \qquad \text{newton (N)}$$

where K is the spring constant, with units newton/m.

The restoring force f(t) of a spring is proportional to the amount x(t) it stretched;



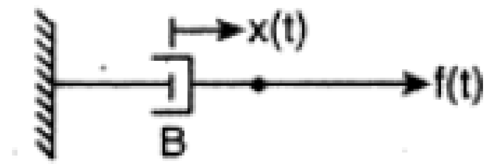


Free-Body Diagram

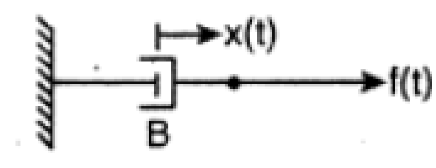
3. Damper (D)

- The damping force f(t) due to viscous friction is proportional to velocity
- Viscous friction is often represented by a dashpot consisting of an oil-filled cylinder and piston.

$$f(t) = B\frac{dx}{dt}$$



Changed into Laplace form: $F = B\ s\ x$



Free-Body Diagram

Damper and Spring not connected to a reference frame

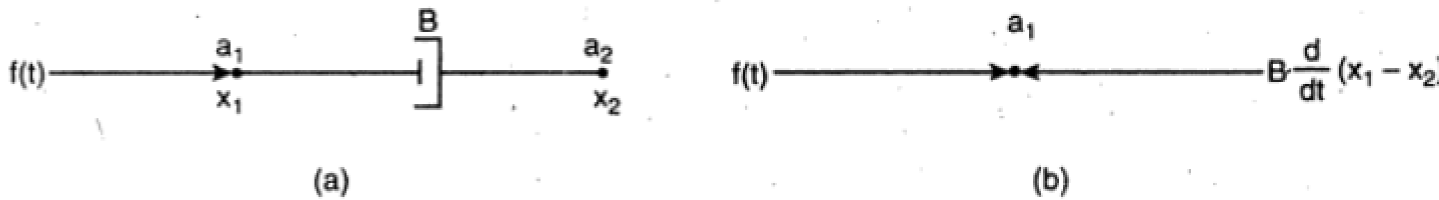


Fig. 2.3 Dash-pot not Connected to Reference Frame

$$f(t) = B\frac{d}{dt}(x_1 - x_2)$$

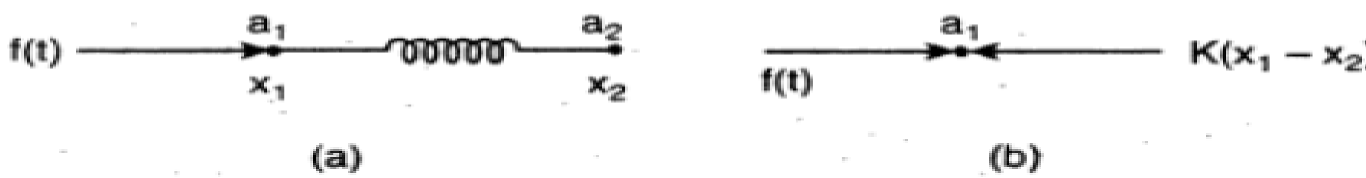
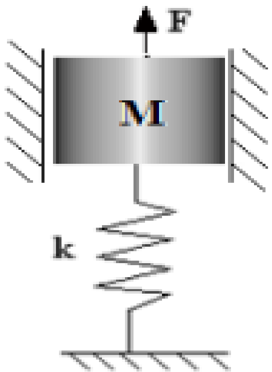
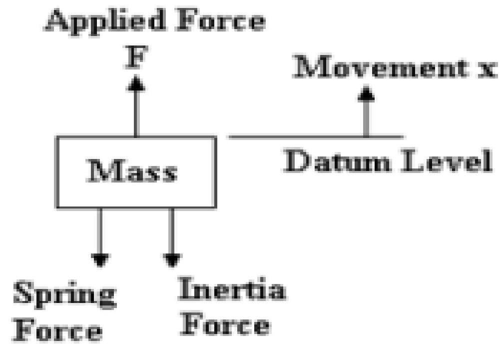


Fig. 2.4 Spring not Connected to Reference Frame

$$f(t) = K\,(x_1 - x_2)$$

MASS AND SPRING SYSTEM





D'Alembert's Principle is that all the forces and moments on the body must add up to zero.

the applied force – spring force – inertia force = 0 (all being a function of time t).

:

$$F(t) - kx(t) - M\, d^2x/dt^2(t) = 0$$

$$F(t) = M\, d^2x/dt^2(t) + kx\,(t)$$

Changing to a function of s we have $F(s) = Ms^2\, x + kx = x\,[Ms^2 + k]$

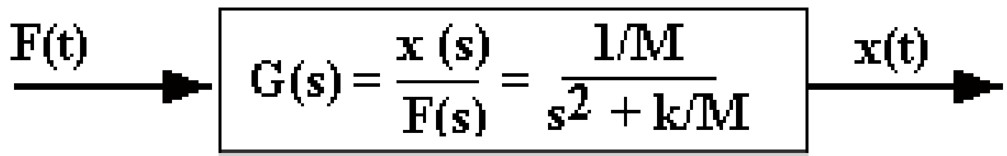
$$x(s) = \frac{F}{Ms^2 + k} = \frac{F(1/M)}{s^2 + k/M}$$

MASS AND SPRING SYSTEM

This may be shown as a transfer function.

$$G(s) = \frac{x(s)}{F(s)} = \frac{1/M}{s^2 + k/M}$$

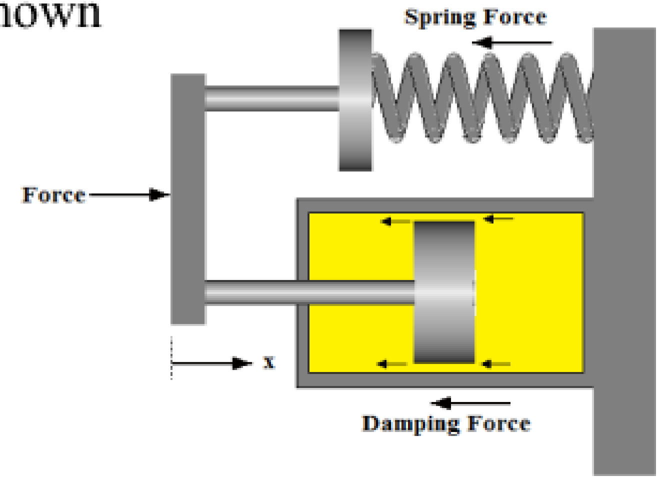
The system block diagram is as shown.



This is a second order system as the highest power of x is 2.

SPRING AND DAMPER

If we have a spring and dashpot in parallel as shown



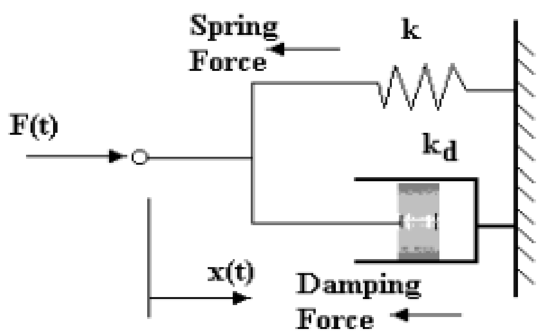
Force balance as a function of time: $F(t) = k\ x + k_d\ dx/dt$

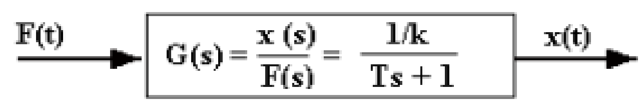
Force balance as a function of s : $F(s) = k\,x + k_d\,s\,x$

Rearrange into a transfer function: $\frac{x}{F}(s) = \frac{1/k}{(k_d/k)s + 1}$

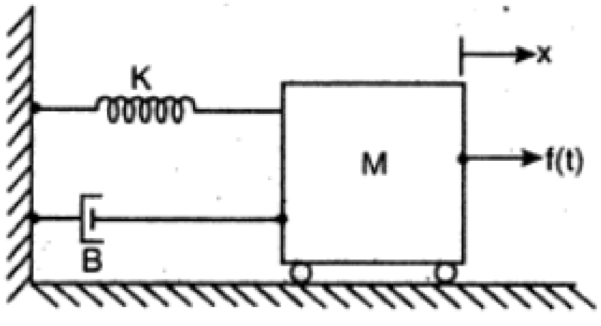
The units of k_d/k are seconds and this is the ***first order time constant for the system***

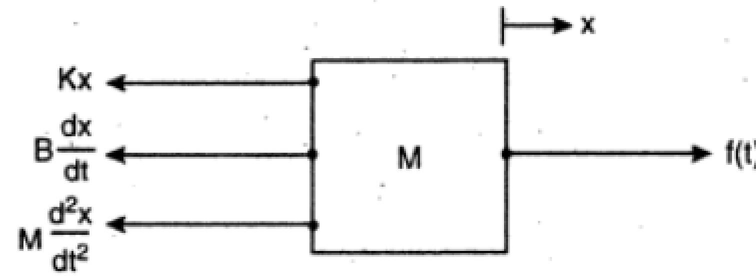
This is the standard first order equation $T = k_d/k$ $\frac{x}{F}(s) = \frac{1/k}{Ts + 1}$





MASS -SPRING - DAMPER SYSTEM





From the free body diagram we easily obtain

the following equation of motion by equating the sum of the forces acting to the right to the sum of the forces acting to the left.

$$f(t) = M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx$$

Taking Laplace transform on both sides of the above equation we get the following transfer function.

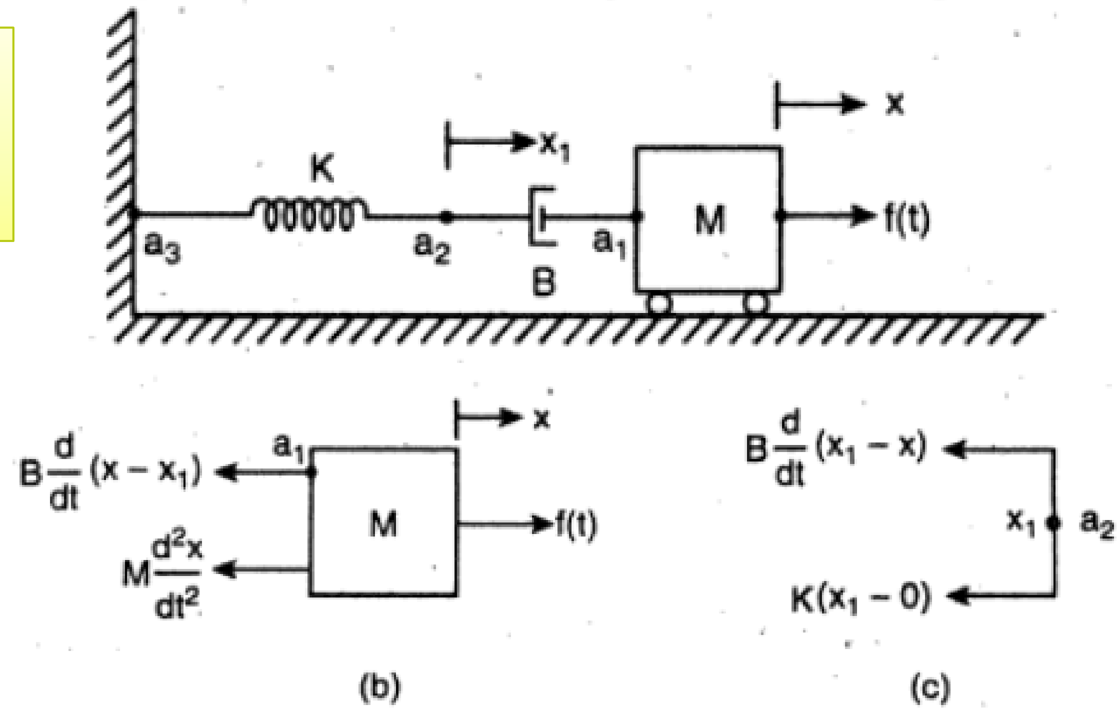
$$F(s) = Ms^2X(s) + BsX(s) + KX(s)$$

or

$$\boxed{\frac{X(s)}{F(s)} = \frac{1}{(Ms^2 + Bs + K)}}$$

MASS -SPRING - DAMPER SYSTEM

When the Mass-Spring-Damper are in Series as shown



Mechanical System with Dash-pot not Connected to Reference Frame

the following dynamic equations are written.

$$f(t) = M\frac{d^2x}{dt^2} + B\frac{d}{dt}(x - x_1)$$

At point a1

$$B\frac{d}{dt}(x_1 - x) + Kx_1 = 0$$

At point a2

MASS -SPRING - DAMPER SYSTEM

Taking Laplace transform on both sides of the above equations the following transfer function model is obtained.

$$F(s) = \left(Ms^2 + Bs\right)X(s) - BsX_1(s)$$

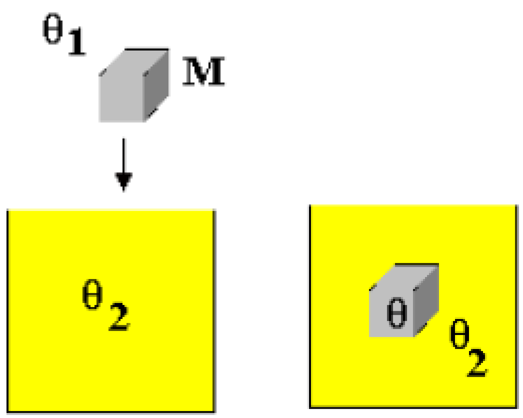
$$\left(Bs + K\right)X_1(s) = BsX(s)$$

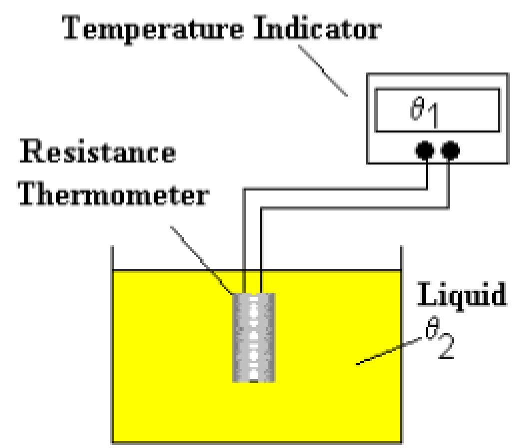
Eliminating $X_1(s)$, the following transfer function is obtained.

$$\frac{X(s)}{F(s)} = \frac{[Bs + K]}{s\left[MBs^2 + MKs + BK\right]}$$

Thermal Systems

- Consider a mass of M Kg at temperature $\Theta_{1.}$
- The mass is submerged in a hot fluid with temperature Θ_2
- The heat Q is transferred into the mass causing its temperature to rise.
- A thermometer is used to monitor the change in the mass temperature with time to see how long it takes for the mass to warm up to the same temperature as the liquid





The laws of heat transfer tell us that the temperature rise is directly proportional to the heat added so:

$$dQ = Mc\, d\theta_1 = C\, d\theta_1$$

c is the specific heat capacity.

C = Mc is the thermal capacitance in Joules/Kelvin.

Thermal Systems

Divide both sides by dt and: $$\frac{dQ}{dt} = \Phi = C\frac{d\theta_1}{dt}$$

The rate of heat transfer into the mass is $\Phi = C\, d\theta_1/dt$ and the rate is governed by

- thermal resistance between the liquid and the mass.
- This obeys a law similar to ohm's law so that:

$\Phi = (\theta_2 - \theta_1)/R$ R is the thermal resistance in Kelvin per Watt.

Equating for Φ we have $$C\frac{d\theta_1}{dt} = \frac{\theta_1 - \theta_2}{R}$$

$$\frac{d\theta_1}{dt} = \frac{\theta_1 - \theta_2}{RC}$$

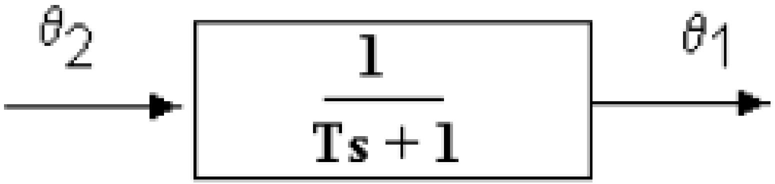
$$\frac{d\theta_1}{dt} + \frac{\theta_1}{RC} = \frac{\theta_2}{RC}$$ RC is a time constant T

$$\frac{d\theta_1}{dt} + \frac{\theta_1}{T} = \frac{\theta_2}{T}$$

Thermal Systems

Changing from a function of time into a function of s we have

$$s\theta_1 + \frac{\theta_1}{T} = \frac{\theta_2}{T} \qquad \theta_1(Ts+1) = \theta_2 \qquad \frac{\theta_1}{\theta_2}(s) = \frac{1}{(Ts+1)}$$



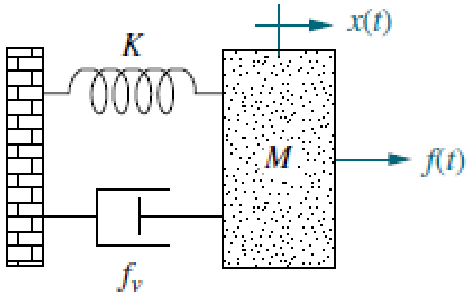
This is another example of standard first order equations

Analogous Electric Circuit

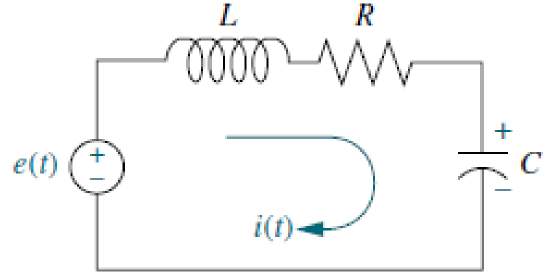
- An electric circuit that is analogous to a system from another discipline is called an **electric circuit analog**.
- The described mechanical systems can be represented by equivalent electric circuits.
- Analogs can be obtained by comparing the equations of motion of a mechanical system, with either electrical mesh or nodal equations.

1. When compared with mesh equations, the resulting electrical circuit is called a **series analog**.
2. When compared with nodal equations, the resulting electrical circuit is called a **parallel analog**.

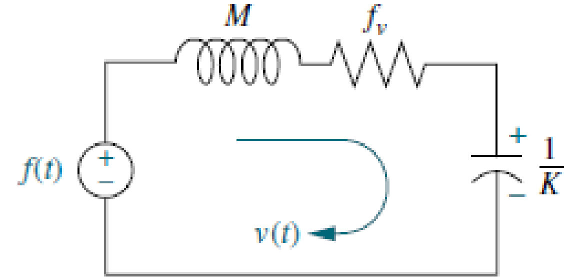
Series Analogous



(a)



(b)



(c)

Equation of motion of the above translational mechanical system is;

$$(Ms^2 + f_v s + K)X(s) = F(s)$$

→ (1)

Kirchhoff's mesh equation for the above simple series RLC network is;

$$\left(Ls + R + \frac{1}{Cs}\right)I(s) = E(s)$$

→ (2)

For a direct analogy b/w Eq (1) & (2), **convert displacement to velocity** by divide and multiply the left-hand side of Eq (1) by s, yielding;

$$\left(Ms + f_v + \frac{K}{s}\right)V(s) = F(s)$$

→ (3)

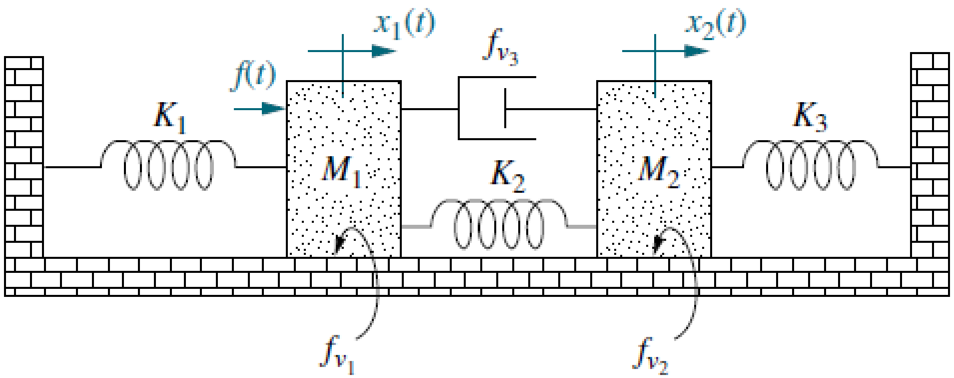
- Comparing Eqs. (2) & (3), we recognize the **sum of impedances** & draw the circuit shown in Figure (c).
- The conversions are summarized in Figure (d).

mass = M	→	inductor	=	M henries
viscous damper = f_v	→	resistor	=	f_v ohms
spring = K	→	capacitor	=	$\frac{1}{K}$ farads
applied force = $f(t)$	→	voltage source	=	$f(t)$
velocity = $v(t)$	→	mesh current	=	$v(t)$

(d)

Example for Converting a Mechanical System to a Series Analog

Example-5: Draw a series analog for the mechanical system.



- The equations of motion in the Laplace transform domain are;

$$[M_1s^2 + (f_{v_1} + f_{v_3})s + (K_1 + K_2)]X_1(s) - (f_{v_3}s + K_2)X_2(s) = F(s) \longrightarrow (1)$$

$$-(f_{v_3}s + K_2)X_1(s) + [M_2s^2 + (f_{v_2} + f_{v_3})s + (K_2 + K_3)]X_2(s) = 0 \longrightarrow (2)$$

- Eqs (1) & (2) are analogous t0 electrical mesh equations after conversion to velocity. Thus,

$$\left[M_1s + (f_{v_1} + f_{v_3}) + \frac{(K_1 + K_2)}{s}\right]V_1(s) - \left(f_{v_3} + \frac{K_2}{s}\right)V_2(s) = F(s) \longrightarrow (3)$$

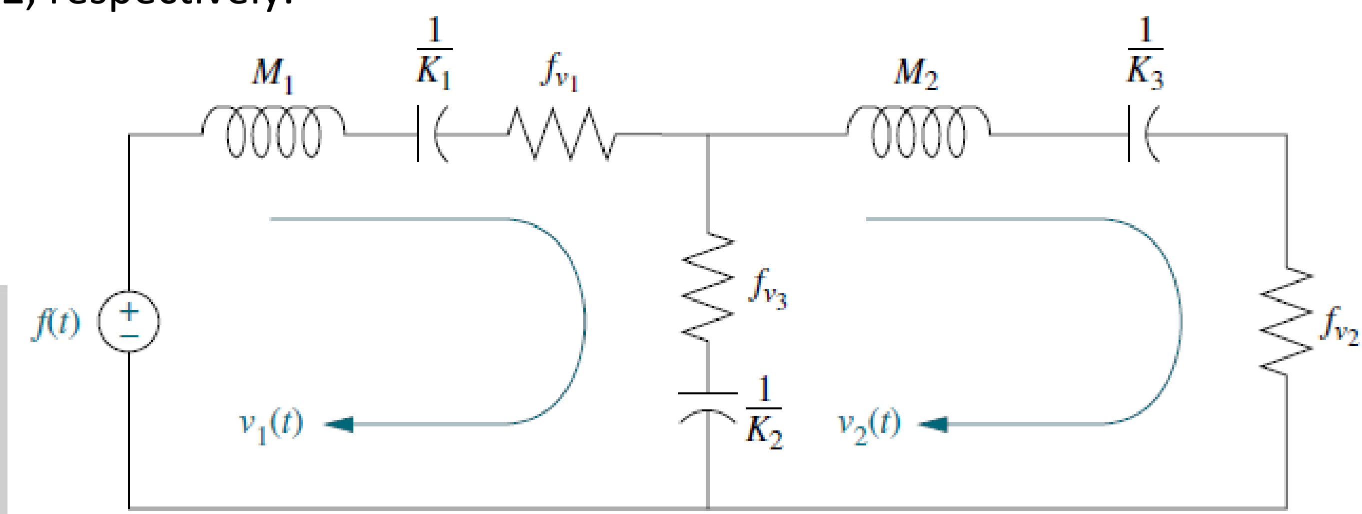
$$-\left(f_{v_3} + \frac{K_2}{s}\right)V_1(s) + \left[M_2s + (f_{v_2} + f_{v_3}) + \frac{(K_2 + K_3)}{s}\right]V_2(s) = 0 \longrightarrow (4)$$

Example-5: Continue.

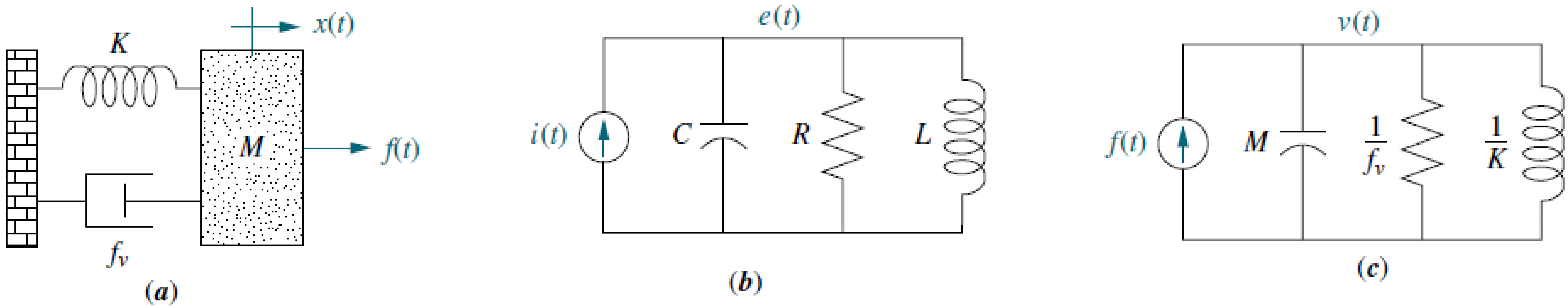
$$\left[M_1 s + (f_{v_1} + f_{v_3}) + \frac{(K_1 + K_2)}{s}\right] V_1(s) - \left(f_{v_3} + \frac{K_2}{s}\right) V_2(s) = F(s) \longrightarrow (3)$$

$$-\left(f_{v_3} + \frac{K_2}{s}\right) V_1(s) + \left[M_2 s + (f_{v_2} + f_{v_3}) + \frac{(K_2 + K_3)}{s}\right] V_2(s) = 0 \longrightarrow (4)$$

- Coefficients represent sums of electrical impedance.
- Mechanical impedances associated withM1 form the first mesh,
- whereas impedances between the two masses are common to the two loops.
- Impedances associated with M2 form the second mesh.
- The result is shown in Figure below, where v1(t) and v2(t) are the velocities of M1 and M2, respectively.



Parallel Analogous



- Equation of motion of the above translational mechanical system is;

$$\left(Ms + f_v + \frac{K}{s}\right)V(s) = F(s) \quad \rightarrow (1)$$

- Kirchhoff's nodal equation for the simple parallel RLC network shown above is;

$$\left(Cs + \frac{1}{R} + \frac{1}{Ls}\right)E(s) = I(s) \quad \rightarrow (2)$$

- Comparing Eqs. (1) & (2), we identify the **sum of admittances** & draw the circuit shown in Figure (c).

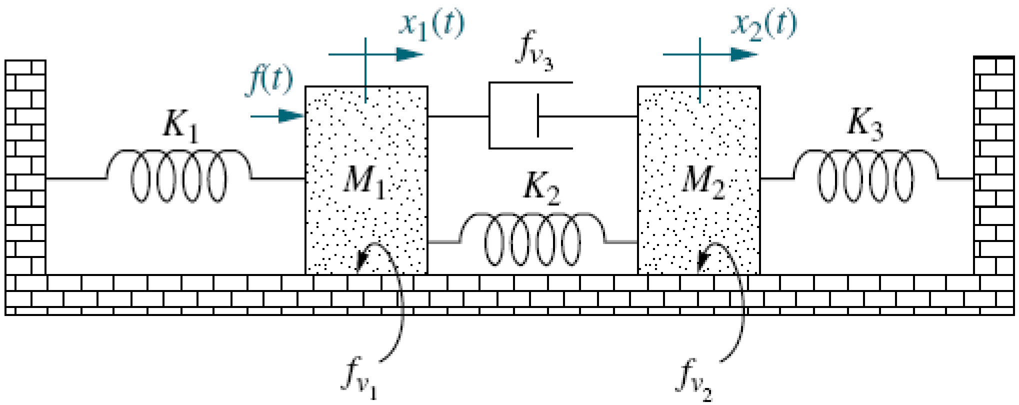
- The conversions are summarized in Figure 2.43(d).

mass = M	$\longrightarrow$	capacitor	$= M$ farads
viscous damper = f_v	$\longrightarrow$	resistor	$= \frac{1}{f_v}$ ohms
spring = K	$\longrightarrow$	inductor	$= \frac{1}{K}$ henries
applied force = $f(t)$	$\longrightarrow$	current source	$= f(t)$
velocity = $v(t)$	$\longrightarrow$	node voltage	$= v(t)$

(d)

Example for Converting a Mechanical System to a Parallel Analog

Example-6: Draw a parallel analog for the mechanical system.



- Equations of motion after conversion to velocity are;

$$\left[M_1 s + (f_{v_1} + f_{v_3}) + \frac{(K_1 + K_2)}{s}\right] V_1(s) - \left(f_{v_3} + \frac{K_2}{s}\right) V_2(s) = F(s) \longrightarrow (1)$$

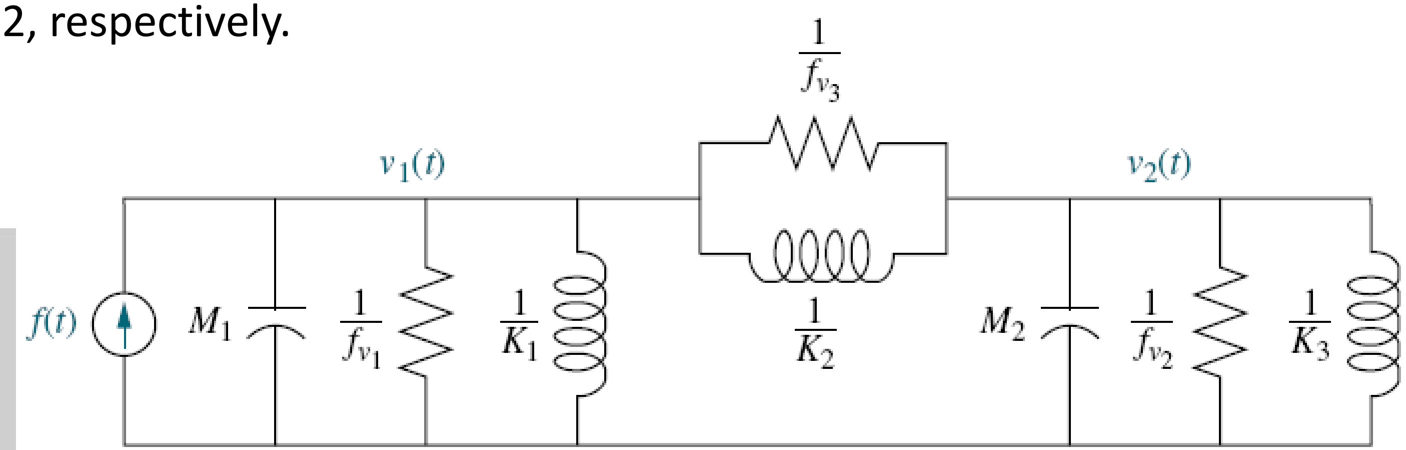
$$-\left(f_{v_3} + \frac{K_2}{s}\right) V_1(s) + \left[M_2 s + (f_{v_2} + f_{v_3}) + \frac{(K_2 + K_3)}{s}\right] V_2(s) = 0 \longrightarrow (2)$$

Example-6: Continue.

$$\left[M_1 s + (f_{v_1} + f_{v_3}) + \frac{(K_1 + K_2)}{s}\right] V_1(s) - \left(f_{v_3} + \frac{K_2}{s}\right) V_2(s) = F(s) \longrightarrow (1)$$

$$-\left(f_{v_3} + \frac{K_2}{s}\right) V_1(s) + \left[M_2 s + (f_{v_2} + f_{v_3}) + \frac{(K_2 + K_3)}{s}\right] V_2(s) = 0 \longrightarrow (2)$$

- The Equation (1) and (2) are also analogous to electrical node equations.
- Coefficients represent sums of electrical admittances.
- Admittances associated with M1 form the elements connected to the first node,
- whereas mechanical admittances b/w the two masses are common to the two nodes.
- Mechanical admittances associated with M2 form the elements connected to the second node.
- The result is shown in the Figure below, where v1(t) and v2(t) are the velocities of M1 and M2, respectively.



	Parallel	Series
Electrical Quantity	**Mechanical Analog I (Force-Current)**	**Mechanical Analog II (Force Voltage)**
Voltage, e	Velocity, v	Force, f
Current, i	Force, f	Velocity, v
Resistance, R	Lubricity, 1/B (Inverse friction) damper const	Friction, B
Capacitance, C	Mass, M	Compliance, 1/K (Inverse spring constant)
Inductance, L	Compliance, 1/K (Inverse spring constant)	Mass, M